

The minimal size of graphs with given pendant-tree connectivity *

Yaping Mao^{1,2†}

¹Department of Mathematics, Qinghai Normal University,

²Key Laboratory of IOT of Qinghai Province,
Xining, Qinghai 810008, China

Abstract

The concept of pendant-tree k -connectivity $\tau_k(G)$ of a graph G , introduced by Hager in 1985, is a generalization of classical vertex-connectivity. Let $f(n, k, \ell)$ be the minimal number of edges of a graph G of order n with $\tau_k(G) = \ell$ ($1 \leq \ell \leq n - k$). In this paper, we give some exact value or sharp bounds of the parameter $f(n, k, \ell)$.

Keywords: connectivity, Steiner tree, packing, pendant-tree connectivity

AMS subject classification 2010: 05C05, 05C40, 05C70.

1 Introduction

A processor network is expressed as a graph, where a node is a processor and an edge is a communication link. Broadcasting is the process of sending a message from the source node to all other nodes in a network. It can be accomplished by message dissemination in such a way that each node repeatedly receives and forwards messages. Some of the nodes and/or links may be faulty. However, multiple copies of messages can be disseminated through disjoint paths. We say that the broadcasting succeeds if all the healthy nodes in the network finally obtain the correct message from the source node within a certain limit of time. A lot of attention has been devoted to fault-tolerant broadcasting in networks [9, 13, 15, 36]. In order to measure the ability of fault-tolerance, the above path structure connecting two nodes are generalized into some tree structures connecting more than two nodes, see [17, 19, 23]. To show these generalizations clearly,

*Supported by the National Science Foundation of China (Nos. 11551001, 11161037, 11461054) and the Science Found of Qinghai Province (No. 2014-ZJ-907).

[†]E-mail: maoyaping@ymail.com

we must state from the connectivity in graph theory. We divide our introduction into the following four subsections to state the motivations and our results of this paper.

1.1 Connectivity and k -connectivity

All graphs considered in this paper are undirected, finite and simple. We refer to the book [1] for graph theoretical notation and terminology not described here. For a graph G , let $V(G)$, $E(G)$, $e(G)$, \bar{G} , $\Delta(G)$ and $\delta(G)$ denote the set of vertices, the set of edges, the size, the complement, the maximum degree and the minimum degree of G , respectively. In the sequel, let $K_{s,t}$, K_n , W_n , C_n , and P_n denote the complete bipartite graph of order $s+t$ with part sizes s and t , complete graph of order n , wheel of order n , cycle of order n , and path of order n , respectively. For any subset X of $V(G)$, let $G[X]$ denote the subgraph induced by X , and $E[X]$ the edge set of $G[X]$. For two subsets X and Y of $V(G)$ we denote by $E_G[X, Y]$ the set of edges of G with one end in X and the other end in Y . If $X = \{x\}$, we simply write $E_G[x, Y]$ for $E_G[\{x\}, Y]$.

Connectivity is one of the most basic concepts of graph-theoretic subjects, both in combinatorial sense and the algorithmic sense. It is well-known that the classical connectivity has two equivalent definitions. The *connectivity* of G , written $\kappa(G)$, is the minimum order of a vertex set $S \subseteq V(G)$ such that $G - S$ is disconnected or has only one vertex. We call this definition the ‘cut’ version definition of connectivity. A well-known theorem of Whitney [38] provides an equivalent definition of connectivity, which can be called the ‘path’ version definition of connectivity. For any two distinct vertices x and y in G , the *local connectivity* $\kappa_G(x, y)$ is the maximum number of internally disjoint paths connecting x and y . Then $\kappa(G) = \min\{\kappa_G(x, y) \mid x, y \in V(G), x \neq y\}$ is defined to be the *connectivity* of G . For connectivity, Oellermann gave a survey paper on this subject; see [31].

Although there are many elegant and powerful results on connectivity in graph theory, the basic notation of classical connectivity may not be general enough to capture some computational settings. So people want to generalize this concept. For the ‘cut’ version definition of connectivity, we find the above minimum vertex set without regard the number of components of $G - S$. Two graphs with the same connectivity may have differing degrees of vulnerability in the sense that the deletion of a vertex cut-set of minimum cardinality from one graph may produce a graph with considerably more components than in the case of the other graph. For example, the star $K_{1,n}$ and the path P_{n+1} ($n \geq 3$) are both trees of order $n+1$ and therefore connectivity 1, but the deletion of a cut-vertex from $K_{1,n}$ produces a graph with n components while the deletion of a cut-vertex from P_{n+1} produces only two components. Chartrand et al. [4] generalized the ‘cut’ version definition of connectivity. For an integer k ($k \geq 2$) and a graph G of order n ($n \geq k$), the *k -connectivity* $\kappa'_k(G)$ is the smallest number of vertices whose removal from G of order

n ($n \geq k$) produces a graph with at least k components or a graph with fewer than k vertices. Thus, for $k = 2$, $\kappa'_2(G) = \kappa(G)$. For more details about k -connectivity, we refer to [4, 16, 31, 32].

1.2 Generalized (edge-)connectivity

The generalized connectivity of a graph G , introduced by Hager [12], is a natural generalization of the ‘path’ version definition of connectivity. For a graph $G = (V, E)$ and a set $S \subseteq V(G)$ of at least two vertices, an S -Steiner tree or a Steiner tree connecting S (or simply, an S -tree) is a such subgraph $T = (V', E')$ of G that is a tree with $S \subseteq V'$. Note that when $|S| = 2$ an S -Steiner tree is just a path connecting the two vertices of S . Two S -Steiner trees T and T' are said to be *internally disjoint* if $E(T) \cap E(T') = \emptyset$ and $V(T) \cap V(T') = S$. For $S \subseteq V(G)$ and $|S| \geq 2$, the *generalized local connectivity* $\kappa_G(S)$ is the maximum number of internally disjoint S -Steiner trees in G , that is, we search for the maximum cardinality of edge-disjoint trees which include S and are vertex disjoint with the exception of S . For an integer k with $2 \leq k \leq n$, *generalized k -connectivity* (or *k -tree-connectivity*) is defined as $\kappa_k(G) = \min\{\kappa_G(S) \mid S \subseteq V(G), |S| = k\}$, that is, $\kappa_k(G)$ is the minimum value of $\kappa_G(S)$ when S runs over all k -subsets of $V(G)$. Clearly, when $|S| = 2$, $\kappa_2(G)$ is nothing new but the connectivity $\kappa(G)$ of G , that is, $\kappa_2(G) = \kappa(G)$, which is the reason why one addresses $\kappa_k(G)$ as the generalized connectivity of G . By convention, for a connected graph G with less than k vertices, we set $\kappa_k(G) = 1$. Set $\kappa_k(G) = 0$ when G is disconnected. Note that the generalized k -connectivity and k -connectivity of a graph are indeed different. Take for example, the graph H_1 obtained from a triangle with vertex set $\{v_1, v_2, v_3\}$ by adding three new vertices u_1, u_2, u_3 and joining v_i to u_i by an edge for $1 \leq i \leq 3$. Then $\kappa_3(H_1) = 1$ but $\kappa'_3(H_1) = 2$. There are many results on the generalized connectivity, see [5, 19, 20, 21, 23, 24, 25, 26, 27, 33].

1.3 Pendant-tree (edge-)connectivity

The concept of pendant-tree connectivity [12] was introduced by Hager in 1985, which is specialization of generalized connectivity (or *k -tree-connectivity*) but a generalization of classical connectivity. For an S -Steiner tree, if the degree of each vertex in S is equal to one, then this tree is called a *pendant S -Steiner tree*. Two pendant S -Steiner trees T and T' are said to be *internally disjoint* if $E(T) \cap E(T') = \emptyset$ and $V(T) \cap V(T') = S$. For $S \subseteq V(G)$ and $|S| \geq 2$, the *pendant-tree local connectivity* $\tau_G(S)$ is the maximum number of internally disjoint pendant S -Steiner trees in G . For an integer k with $2 \leq k \leq n$, *pendant-tree k -connectivity* is defined as $\tau_k(G) = \min\{\tau_G(S) \mid S \subseteq V(G), |S| = k\}$. When $k = 2$, $\tau_2(G) = \tau(G)$ is just the connectivity of a graph G . For more details on pendant-tree

connectivity, we refer to [12, 28]. Clearly, we have

$$\begin{cases} \tau_k(G) = \kappa_k(G), & \text{for } k = 1, 2; \\ \tau_k(G) \leq \kappa_k(G), & \text{for } k \geq 3. \end{cases}$$

The relation between pendant-tree connectivity and generalized connectivity are shown in the following Table 2.

| | Pendant tree-connectivity | Generalized connectivity |
|----------------------|--|---|
| Vertex subset | $S \subseteq V(G)$ ($ S \geq 2$) | $S \subseteq V(G)$ ($ S \geq 2$) |
| Set of Steiner trees | $\begin{cases} \mathcal{T}_S = \{T_1, T_2, \dots, T_\ell\} \\ S \subseteq V(T_i), \\ d_{T_i}(v) = 1 \text{ for every } v \in S \\ E(T_i) \cap E(T_j) = \emptyset, \end{cases}$ | $\begin{cases} \mathcal{T}_S = \{T_1, T_2, \dots, T_\ell\} \\ S \subseteq V(T_i), \\ E(T_i) \cap E(T_j) = \emptyset, \end{cases}$ |
| Local parameter | $\tau(S) = \max \mathcal{T}_S $ | $\kappa(S) = \max \mathcal{T}_S $ |
| Global parameter | $\tau_k(G) = \min_{S \subseteq V(G), S =k} \tau(S)$ | $\kappa_k(G) = \min_{S \subseteq V(G), S =k} \kappa(S)$ |

Table 2. Two kinds of tree-connectivities

The following two observations are easily seen.

Observation 1.1 *If G is a connected graph, then $\tau_k(G) \leq \mu_k(G) \leq \delta(G)$.*

Observation 1.2 *If H is a spanning subgraph of G , then $\tau_k(H) \leq \tau_k(G)$.*

In [12], Hager derived the following results.

Lemma 1.1 [12] *Let G be a graph. If $\tau_k(G) \geq \ell$, then $\delta(G) \geq k + \ell - 1$.*

Lemma 1.2 [12] *Let G be a graph. If $\tau_k(G) \geq \ell$, then $\kappa(G) \geq k + \ell - 2$.*

Lemma 1.3 [12] *Let k, n be two integers with $3 \leq k \leq n$, and let K_n be a complete graph of order n . Then*

$$\tau_k(K_n) = n - k.$$

Lemma 1.4 [12] *Let $K_{r,s}$ be a complete bipartite graph with $r + s$ vertices. Then*

$$\tau_k(K_{r,s}) = \max\{\min\{r - k + 1, s - k + 1\}, 0\}.$$

As a natural counterpart of the pendant-tree k -connectivity, we introduced the concept of pendant-tree k -edge-connectivity. For $S \subseteq V(G)$ and $|S| \geq 2$, the *pendant-tree local edge-connectivity* $\mu(S)$ is the maximum number of edge-disjoint pendant S -Steiner trees in G . For an integer k with $2 \leq k \leq n$, the *pendant-tree k -edge-connectivity* $\mu_k(G)$ of G is then defined as $\mu_k(G) = \min\{\mu(S) \mid S \subseteq V(G) \text{ and } |S| = k\}$. It is also clear that when $|S| = 2$, $\mu_2(G)$ is just the standard edge-connectivity $\lambda(G)$ of G , that is, $\mu_2(G) = \lambda(G)$.

1.4 Application background and our results

In addition to being a natural combinatorial measure, both the pendant-tree connectivity and the generalized connectivity can be motivated by its interesting interpretation in practice. For example, suppose that G represents a network. If one considers to connect a pair of vertices of G , then a path is used to connect them. However, if one wants to connect a set S of vertices of G with $|S| \geq 3$, then a tree has to be used to connect them. This kind of tree with minimum order for connecting a set of vertices is usually called a Steiner tree, and popularly used in the physical design of VLSI (see [10, 11, 34]) and computer communication networks (see [8]). Usually, one wants to consider how tough a network can be, for the connection of a set of vertices. Then, the number of totally independent ways to connect them is a measure for this purpose. The generalized k -connectivity can serve for measuring the capability of a network G to connect any k vertices in G .

Let $f(n, k, \ell)$ be the minimal number of edges of a graph G of order n with $\tau_k(G) = \ell$ ($1 \leq \ell \leq n - k$). It is not easy to determine the exact value of the parameter $f(n, k, \ell)$ for a general k ($3 \leq k \leq n$) and a general ℓ ($1 \leq \ell \leq n - k$).

In Section 2, we obtain the following result for general k .

Theorem 1.1 *Let n, k be two integers with $3 \leq k \leq n$ and $n \geq 15$. Then*

- (1) $f(n, k, n - k) = \binom{n}{2}$;
- (2) $f(n, k, n - k - 1) = \binom{n}{2} - 2$;
- (3) $f(n, k, 0) = n - 1$;
- (4) $f(n, k, 1) = \lceil \frac{kn}{2} \rceil$;
- (5) For $1 \leq \ell \leq \frac{n}{2} - k + 1$, we have

$$\left\lceil \frac{1}{2}(k + \ell - 1)n \right\rceil \leq f(n, k, \ell) \leq (k + \ell - 1)(n - k - \ell + 1);$$

- (6) For $\frac{n}{2} - k + 2 \leq \ell \leq n - k - 2$, we have

$$\left\lceil \frac{1}{2}(k + \ell - 1)n \right\rceil \leq f(n, k, \ell) \leq (k + \ell - 1)(n - k - \ell + 1) + \binom{k + \ell - 1}{2}.$$

Moreover, the bounds are sharp.

For $k = n, n-1, n-2, n-3, 3$, we get the following results in Section 3.

Theorem 1.2 *Let n, k be two integers with $3 \leq k \leq n$ and $n \geq 15$. Then*

- (1) $f(n, n, 0) = n - 1$;
- (2) $f(n, n-1, 1) = \binom{n}{2}$, $f(n, n-1, 0) = n - 1$;
- (3) $f(n, n-2, 2) = \binom{n}{2}$, $f(n, n-2, 1) = \binom{n}{2} - 2$, $f(n, n-2, 0) = n - 1$;
- (4) $f(n, n-3, 3) = \binom{n}{2}$, $f(n, n-3, 2) = \binom{n}{2} - 2$, $f(n, n-3, 1) = \binom{n}{2} - n$; $f(n, n-3, 0) = n - 1$.

Theorem 1.3 *Let n be an integer with $n \geq 10$. Then*

- (1) $f(n, 3, 0) = n - 1$;
- (2) $f(n, 3, 1) = \lceil \frac{3n}{2} \rceil$;
- (3) $f(n, 3, 2) = 2n$ for $n = pq$ and $p, q \geq 3$; $2n \leq f(n, 3, 2) \leq \frac{5n}{2}$ for $n = 2p$; $2n \leq f(n, 3, 2) \leq 4n - 16$ if n is a prime number.
- (4) $\left\lceil \frac{(\ell+2)n}{2} \right\rceil \leq f(n, 3, \ell) \leq (\ell+1)(n-\ell) + 1$ for $3 \leq \ell \leq \frac{n-4}{3}$;
- (5) $\left\lceil \frac{(\ell+2)n}{2} \right\rceil \leq f(n, 3, \ell) \leq \left\lfloor \frac{n}{\ell+1} \right\rfloor (\ell+1)^2 + (r+1)(\ell+1) + r - 1$ for $\frac{n-1}{3} \leq \ell \leq \frac{n-r-2}{2}$ where $n \equiv r \pmod{\ell}$ and $r \geq 2$; $\left\lceil \frac{(\ell+2)n}{2} \right\rceil \leq f(n, 3, \ell) \leq (\ell+2)(n-\ell-2)$ for $\frac{n-r-2}{2} \leq \ell \leq \frac{n-4}{2}$, or $\frac{n-1}{3} \leq \ell \leq \frac{n-r-2}{2}$ where $n \equiv r \pmod{\ell}$ and $r = 0, 1$;
- (6) $\left\lceil \frac{(\ell+2)n}{2} \right\rceil \leq f(n, 3, \ell) \leq (\ell+2)(n-\ell-2) + \binom{\ell+2}{2}$ for $\frac{n-2}{2} \leq \ell \leq n-6$;
- (7) $f(n, 3, n-5) = \binom{n}{2} - \lfloor \frac{n-4}{2} \rfloor$;
- (8) $f(n, 3, n-4) = \binom{n}{2} - 2$;
- (9) $f(n, 3, n-3) = \binom{n}{2}$.

2 For general k and ℓ

Given a vertex x and a set U of vertices, an (x, U) -fan is a set of paths from x to U such that any two of them share only the vertex x . The size of an (x, U) -fan is the number of internally disjoint paths from x to U .

Lemma 2.1 (Fan Lemma, [37], p-170) *A graph is k -connected if and only if it has at least $k+1$ vertices and, for every choice of x, U with $|U| \geq k$, it has an (x, U) -fan of size k .*

Corollary 2.1 *Let G be a graph with $\kappa(G) = k$. Then $\tau_k(G) \geq 1$.*

Proof. Since $\kappa(G) = k$, it follows from Lemma 2.1 that for any $S \subseteq V(G)$ and $|S| = k$, there exist an (x, S) -fan, where $x \in V(G) - S$. So there is a pendant S -Steiner tree in G , and hence $\tau(S) \geq 1$. From the arbitrariness of S , we have $\tau_k(G) \geq 1$, as desired. \blacksquare

In [29], Mao and Lai obtained the following upper bound of $\tau_k(G)$.

Lemma 2.2 [29] *Let G be a connected graph of order $n \geq 6$. Let k be an integer with $3 \leq k \leq n$. Then*

$$\tau_k(G) \leq \delta(G) - k + 1.$$

Moreover, the upper bound is sharp.

The Harary graph $H_{n,d}$ is constructed by arranging the n vertices in a circular order and spreading the d edges around the boundary in a nice way, keeping the chords as short as possible. Harary graph $H_{n,d}$ is a d -connected graph on n vertices, and the structure of $H_{n,d}$ depends on the parities of d and n ; see [37].

Case 1: d even. Let $d = 2r$. Then $H_{n,2r}$ is constructed as follows. It has vertices $0, 1, \dots, n-1$ and two vertices i and j are jointed if $i - r \leq j \leq i + r$ (where addition is taken modulo n).

Case 2: d odd, n even. Let $d = 2r + 1$. Then $H_{n,2r+1}$ is constructed by first drawing $H_{n,2r}$ and then adding edges joining vertex i to vertex $i + \frac{n}{2}$ for $1 \leq i \leq \frac{n}{2}$.

Case 3: d odd, n even. Let $d = 2r + 1$. Then $H_{n,2r+1}$ is constructed by first drawing $H_{n,2r}$ and then adding edges joining vertex 0 to vertices $\frac{n-1}{2}$ and $\frac{n+1}{2}$ and i to vertex $i + \frac{n+1}{2}$ for $1 \leq i \leq \frac{n-1}{2}$.

Lemma 2.3 [37] *Let $H_{n,k}$ be the Harary graph of order $n \geq 6$. Then*

$$\kappa(H_{n,k}) = k, \quad e(H_{n,k}) = \left\lceil \frac{kn}{2} \right\rceil.$$

For general k and ℓ , we give a sharp lower bound of $f(n, k, \ell)$.

Proposition 2.1 *Let n, k be two integers with $3 \leq k \leq n$ and $n \geq 15$. Then*

$$f(n, k, \ell) \geq \left\lceil \frac{1}{2}(k + \ell - 1)n \right\rceil$$

for $1 \leq \ell \leq n - k - 2$. Moreover, the bound is sharp.

Proof. Since $\tau_k(G) = \ell$ ($1 \leq \ell \leq n - k - 2$), it follows from Lemma 2.2 that $\delta(G) \geq k + \ell - 1$. Denote by X the set of vertices of degree $k + \ell - 1$ in G . Set $Y = V(G) \setminus X$. Let m', m'' be the number of edges of $G[X], G[Y]$, respectively. Then

$$|E_G[X, Y]| = (k + \ell - 1)|X| - 2m' \tag{1}$$

and

$$\begin{aligned}
e(G) &= e(G[X]) + e(G[Y]) + |E_G[X, Y]| \\
&= m' + m'' + (k + \ell - 1)|X| - 2m' \\
&= (k + \ell - 1)|X| + (m'' - m').
\end{aligned} \tag{2}$$

Since every vertex in Y has degree at least $k + \ell$ in G , it follow that

$$\begin{aligned}
\sum_{v \in Y} d(v) &= 2m'' + |E_G[X, Y]| \\
&= 2m'' + (k + \ell - 1)|X| - 2m' \\
&\geq (k + \ell)|Y|,
\end{aligned}$$

and hence

$$m'' - m' \geq \frac{1}{2}(k + \ell - 1)(|Y| - |X|) + \frac{1}{2}|Y|. \tag{3}$$

From (1), (2), (3), we have

$$e(G) = \frac{1}{2}(k + \ell - 1)(|Y| + |X|) + \frac{1}{2}|Y| \geq \frac{1}{2}(k + \ell - 1)n.$$

Since the number of edges is an integer, it follows that

$$e(G) \geq \left\lceil \frac{1}{2}(k + \ell - 1)n \right\rceil,$$

as desired. ■

To show the sharpness of the lower bound, we consider the following example.

Example 1. Let $H_{n,k}$ be the Harary graph of order $n \geq 6$. Then $\kappa(H_{n,k}) = \delta(H_{n,k}) = k$. From Lemma 2.2, we have $\tau_k(H_{n,k}) \leq \delta(H_{n,k}) - k + 1 = 1$. From Corollary 2.1, we have $\tau_k(H_{n,k}) \geq 1$, and hence $\tau_k(H_{n,k}) = 1$. From Lemma 2.3, $e(H_{n,k}) = \left\lceil \frac{kn}{2} \right\rceil = f(n, k, 1)$. So the lower bound is sharp for $\ell = 1$.

The following corollary is immediate from Proposition 2.1 and Example 1.

Corollary 2.2 *Let n, k be two integers with $3 \leq k \leq n$ and $n \geq 5$. Then*

$$f(n, k, 1) = \left\lceil \frac{kn}{2} \right\rceil.$$

For general k and ℓ ($1 \leq \ell \leq \frac{n}{2} - k + 1$), we give a sharp upper bound of $f(n, k, \ell)$.

Proposition 2.2 *Let n, k, ℓ be three integers with $3 \leq k \leq n$ and $1 \leq \ell \leq \frac{n}{2} - k + 1$. Then*

$$f(n, k, \ell) \leq (k + \ell - 1)(n - k - \ell + 1).$$

Moreover, the bound is sharp.

Proof. Let $G = K_{k+\ell-1, n-k-\ell+1}$. Since $1 \leq \ell \leq \frac{n}{2} - k + 1$, it follows that $n - k - \ell + 1 \geq k + \ell - 1$. From Lemma 1.4, we have $\tau_k(G) = \ell$. So $f(n, k, \ell) \leq (k + \ell - 1)(n - k - \ell + 1)$, as desired. \blacksquare

The *join* or *complete product* $G \vee H$ of two disjoint graphs G and H is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv \mid u \in V(G), v \in V(H)\}$. For classical connectivity, the following result is well-known.

Lemma 2.4 [1] *Let G and H be two graphs. Then*

$$\kappa(G \vee H) = \min\{|V(G)| + \kappa(H), |V(H)| + \kappa(G)\}.$$

For general k and ℓ ($\frac{n}{2} - k + 2 \leq \ell \leq n - k$), we can also give a sharp upper bound of $f(n, k, \ell)$.

Proposition 2.3 *Let n, k, ℓ be three integers with $3 \leq k \leq n$ and $\frac{n}{2} - k + 2 \leq \ell \leq n - k$. Then*

$$f(n, k, \ell) \leq (k + \ell - 1)(n - k - \ell + 1) + \binom{k + \ell - 1}{2}.$$

Moreover, the bound is sharp.

Proof. Let $G = K_{k+\ell-1} \vee (n - k - \ell + 1)K_1$. Since $\delta(G) = k + \ell - 1$, it follows from Lemma 2.2 that $\tau_k(G) \leq \delta(G) - k + 1 = \ell$. It suffices to show that $\tau_k(G) \geq \ell$. Let $V(K_{k+\ell-1}) = \{y_1, y_2, \dots, y_{k+\ell-1}\}$ and $X = V(G) - V(K_{k+\ell-1}) = \{x_1, x_2, \dots, x_{n-k-\ell+1}\}$. For any $S \subseteq V(G)$ and $|S| = k$, if $S \subseteq X$, then the trees induced by the edges in $E_G[y_i, S]$ ($1 \leq i \leq k + \ell - 1$) are $k + \ell - 1$ internally disjoint pendant S -Steiner trees, and hence $\tau(S) \geq k + \ell - 1$. If $S \subseteq Y$, then there are $\ell - 1$ internally disjoint pendant S -Steiner trees in $G[Y]$. These trees and the trees induced by the edges in $E_G[x_i, S]$ ($1 \leq i \leq n - k - \ell + 1$) are $n - k$ internally disjoint pendant S -Steiner trees, and hence $\tau(S) \geq n - k \geq \ell$. Suppose $S \cap X \neq \emptyset$ and $S \cap V(K_{k+\ell-1}) \neq \emptyset$. Without loss of generality, let $S \cap X = \{x_1, x_2, \dots, x_r\}$ and $S \cap V(K_{k+\ell-1}) = \{y_1, y_2, \dots, y_{k-r}\}$. Then the trees induced by the edges in $\{x_i y_j \mid 1 \leq i \leq r, k - r + 1 \leq j \leq k + \ell - 1\} \cup \{y_i y_j \mid 1 \leq i \leq k - r, k - r + 1 \leq j \leq k + \ell - 1\}$ are $\ell - 1 + r \geq \ell$ internally disjoint pendant S -Steiner trees, and hence $\tau(S) \geq n - 1$. We conclude that $\tau(S) \geq n - 1$ for any $S \subseteq V(G)$ and $|S| = k$. From the arbitrariness of S , we have $\tau_k(G) \geq \ell$, and hence $\tau_k(G) = \ell$. So $f(n, k, \ell) \leq (k + \ell - 1)(n - k - \ell + 1) + \binom{k + \ell - 1}{2}$. \blacksquare

In [28], Mao and Lai characterized graphs with $\tau_k(G) = n - k, n - k - 1, n - k - 2$, respectively.

Lemma 2.5 [28] *Let k, n be two integers with $3 \leq k \leq n$ and $n \geq 4$, and let G be a connected graph. Then $\tau_k(G) = n - k$ if and only if G is a complete graph of order n .*

Lemma 2.6 [28] *Let k, n be two integers with $3 \leq k \leq n$ and $n \geq 7$, and let G be a connected graph. Then $\tau_k(G) = n - k - 1$ if and only if $\bar{G} = rK_2 \cup (n - 2r)K_1$ ($r = 1, 2$).*

From $\ell = n - k, n - k - 1, 0$, we can derive the exact value of $f(n, k, \ell)$.

Proposition 2.4 *Let n, k be two integers with $3 \leq k \leq n$ and $n \geq 15$. Then*

- (i) $f(n, k, n - k) = \binom{n}{2}$;
- (ii) $f(n, k, n - k - 1) = \binom{n}{2} - 2$;
- (iii) $f(n, k, 0) = n - 1$.

Proof. (i) From Lemma 2.5, $\tau_k(G) = n - k$ if and only if G is a complete graph of order n . So $f(n, k, n - k) = \binom{n}{2}$.

(ii) From Lemma 2.6, $\tau_k(G) = n - k - 1$ if and only if $\bar{G} = rK_2 \cup (n - 2r)K_1$ ($r = 1, 2$). Then $f(n, k, n - k - 1) = \binom{n}{2} - 2$.

(iii) The tree T_n with n vertices is the graph such that $\tau_k(T_n) = 0$ with the minimal number of edges. So $f(n, k, 1) = n - 1$. ■

The results in Theorem 1.1 follow from Propositions 2.1, 2.2, 2.3 and 2.4.

3 For fixed k and general ℓ

The *Cartesian product* $G \square H$ of two graphs G and H , is the graph with vertex set $V(G) \times V(H)$, in which two vertices (u, v) and (u', v') are adjacent if and only if $u = u'$ and $(v, v') \in E(H)$, or $v = v'$ and $(u, u') \in E(G)$. Clearly, $|E(G \square H)| = |E(H)||V(G)| + |E(G)||V(H)|$.

The *lexicographic product* $G \circ H$ of graphs G and H has the vertex set $V(G \circ H) = V(G) \times V(H)$, and two vertices $(u, v), (u', v')$ are adjacent if $uu' \in E(G)$, or if $u = u'$ and $vv' \in E(H)$. It is easy to see that $|E(G \circ H)| = |E(H)||V(G)| + |E(G)||V(H)|^2$.

In this section, we study $f(n, k, \ell)$ for $k = n, n - 1, n - 2, n - 3, 3$ and general ℓ .

3.1 For $n - 3 \leq k \leq n$ and general ℓ

Mao and Lai obtained the following results.

Lemma 3.1 [28] *Let G be a graph of order n . Then $\tau_n(G) = 0$ if and only if G is a graph of order n .*

Lemma 3.2 [28] *Let G be a connected graph of order n . Then*

- (1) $\tau_{n-1}(G) = 1$ if and only if G is a complete graph of order n .
- (2) $\tau_{n-1}(G) = 0$ if and only if G is not a complete graph of order n .

Lemma 3.3 [28] *Let G be a connected graph of order n . Then*

- (1) $\tau_{n-2}(G) = 2$ if and only if G is a complete graph of order n .
- (2) $\tau_{n-2}(G) = 1$ if and only if $G = K_n \setminus M$ and $1 \leq |M| \leq 2$, where M is a matching of K_n for $n \geq 7$.
- (3) $\tau_{n-2}(G) = 0$ if and only if G is one of the other graphs.

Graphs with $\tau_{n-3}(G) = \ell$ ($0 \leq \ell \leq 3$) can be also characterized.

Proposition 3.1 *Let G be a connected graph of order $n \geq 9$. Then*

- (1) $\tau_{n-3}(G) = 3$ if and only if G is a complete graph of order n .
- (2) $\tau_{n-3}(G) = 2$ if and only if $G = K_n \setminus M$ and $1 \leq |M| \leq 2$, where M is a matching of K_n .
- (3) $\tau_{n-3}(G) = 1$ if and only if $1 \leq \Delta(\bar{G}) \leq 2$, and $|M| \geq 3$ if M is a matching of \bar{G} .
- (4) $\tau_{n-3}(G) = 0$ if and only if G is one of the other graphs.

Proof. From Lemmas 2.5 and 2.6, (1) and (2) is true. We only need to show that (3) is true. Suppose that $\tau_{n-3}(G) = 1$. We claim that $1 \leq \Delta(\bar{G}) \leq 2$. Assume, to the contrary, that $\Delta(\bar{G}) \geq 3$. Then there exist four vertices u, v, w, u_1 such that $u_1u, u_1v, u_1w \notin E(G)$. We choose $S \subseteq V(G)$ and $|S| = n - 3$ such that $V(G) - S = \{u, v, w\}$. Then $u_1 \in S$. Observe that there is no pendant S -Steiner tree in G . Therefore, $\tau_{n-3}(G) = 0$, a contradiction. So $1 \leq \Delta(\bar{G}) \leq 2$. From (2), if M is a matching of \bar{G} , then $|M| \geq 3$.

Conversely, we suppose $1 \leq \Delta(\bar{G}) \leq 2$, and $|M| \geq 3$ if M is a matching of \bar{G} . For any $S \subseteq V(G)$ and $|S| = n - 3$, there exist three vertices, say u, v, w , such that $\bar{S} = V(G) - S = \{u, v, w\}$. Since $1 \leq \Delta(\bar{G}) \leq 2$, it follows that $1 \leq d_{\bar{G}}(u), d_{\bar{G}}(v), d_{\bar{G}}(w) \leq 2$. Set $S = \{u_1, u_2, \dots, u_{n-3}\}$. If $d_{\bar{G}[\bar{S}]}(u) = 2$, then $uu_i \in E(G)$ ($1 \leq i \leq n - 3$), and hence the tree induced by the edges in $E_G[u, S]$ is a pendant S -Steiner tree, and hence $\tau(S) \geq 1$. The same is true when $d_{\bar{G}[\bar{S}]}(v) = 2$ or $d_{\bar{G}[\bar{S}]}(w) = 2$. From now on, we suppose that $d_{\bar{G}[\bar{S}]}(u) \leq 1$, $d_{\bar{G}[\bar{S}]}(v) \leq 1$ and $d_{\bar{G}[\bar{S}]}(w) \leq 1$. Clearly, we have $G[\bar{S}] = P_3$ or $G[\bar{S}] = K_3$. Suppose $G[\bar{S}] = P_3$. Without loss of generality, let $uv, uw \in E(G)$. Since $1 \leq \Delta(\bar{G}) \leq 2$, there are at most two vertices in S , say u_1, u_2 , such that $uu_1, uu_2 \notin E(G)$. Since $1 \leq \Delta(\bar{G}) \leq 2$, it follows that $u_iv \in E(G)$ or $u_iw \in E(G)$ for $i = 1, 2$. If $u_1w, u_2w \in E(G)$, then the tree induced by the edges in $\{uu_i \mid 3 \leq i \leq n - 3\} \cup \{u_1w, u_2w\}$ is a pendant S -Steiner tree, and hence $\tau(S) \geq 1$. If $u_1v, u_2w \in E(G)$, then the tree induced by the edges in $\{uu_i \mid 3 \leq i \leq n - 3\} \cup \{u_1v, u_2w, uv, vw\}$ is a pendant S -Steiner tree, and hence

$\tau(S) \geq 1$. From the arbitrariness of S , we have $\tau_{n-3}(G) \geq 1$. Since $|M| \geq 3$ if M is a matching of \bar{G} , it follows that $\tau_{n-3}(G) \leq 1$. So $\tau_{n-3}(G) = 1$. \blacksquare

Theorem 1.2 is immediate from Lemmas 3.1, 3.2, 3.3 and Proposition 3.1.

3.2 For $k = 3$ and general ℓ

In [35], Špacapan obtained the following result.

Lemma 3.4 [35] *Let G and H be two nontrivial graphs. Then*

$$\kappa(G \square H) = \min\{\kappa(G)|V(H)|, \kappa(H)|V(G)|, \delta(G) + \delta(H)\}.$$

For $k = 3$ and $\ell = 2$, we have the following.

Proposition 3.2 *Let n be an integer with $n \geq 10$.*

- (1) *If $n = pq$ and $p, q \geq 3$, then $f(n, 3, 2) = 2n$;*
- (2) *If $n = 2p$, then $2n \leq f(n, 3, 2) \leq \frac{5n-8}{2}$;*
- (3) *If n is a prime number, then $2n \leq f(n, 3, 2) \leq 4n - 16$.*

Proof. (1) If $n = pq$ and $p, q \geq 3$, then we consider the graph $G = C_p \square C_q$. From Lemma 3.4, $\kappa(G) = \delta(G) = 4$. From Lemma 2.2, we have $\tau_3(G) \leq \delta(G) - 2 = 2$. We only need to show that $\tau_3(G) \geq 2$. Let $C_{p,1}, C_{p,2}, \dots, C_{p,q}$ be cycles in G corresponding to C_p , and let $C'_{1,q}, C'_{2,q}, \dots, C'_{p,q}$ be cycles in G corresponding to C_q . For any $S \subseteq V(G)$ and $|S| = 3$, it suffices to show $\tau(S) \geq 2$. Let $S = \{x, y, z\}$.

If there exists some cycle $C_{p,i}$ such that $|S \cap V(C_{p,i})| = 3$, then $x, y, z \in V(C_{p,i})$. Let x', y', z' be the vertices in $C_{p,i-1}$ corresponding to x, y, z in $C_{p,i}$, and let x'', y'', z'' be the vertices in $C_{p,i+1}$ corresponding to x, y, z in $C_{p,i}$, respectively. Then the subgraph induced by the edges in $E(C_{p,i-1}) \cup \{xx', yy', zz'\}$ contains a pendant S -Steiner tree, and the subgraph induced by the edges in $E(C_{p,i+1}) \cup \{xx'', yy'', zz''\}$ contains a pendant S -Steiner tree. Note that these trees are internally disjoint. Then $\tau(S) \geq 2$.

Suppose that there exists some cycle $C_{p,i}$ such that $|S \cap V(C_{p,i})| = 2$. Then there exists another cycle $C_{p,j}$ such that $|S \cap V(C_{p,j})| = 1$. Without loss of generality, let $|S \cap V(C_{p,1})| = 2$ and $|S \cap V(C_{p,2})| = 1$. Then $x, y \in V(C_{p,1})$ and $z \in V(C_{p,2})$. Let x', y' be the vertices corresponding to x, y in $C_{p,1}$, and let z' be the vertices corresponding to z in $C_{p,2}$. Suppose $z' \notin \{x, y\}$. Then the vertices x, y, z' divide the cycle $C_{p,1}$ into three paths, say P_1, P_2, P_3 . Similarly, the vertices x', y', z divide the cycle $C_{p,2}$ into three paths, say Q_1, Q_2, Q_3 . Then the tree induced by the edges in $E(P_2) \cup E(P_3) \cup \{zz'\}$ and the tree induced by the edges in $E(Q_1) \cup E(Q_3) \cup \{xx', yy'\}$ are two internally disjoint pendant

S -Steiner trees in G , and hence $\tau(S) \geq 2$. Suppose $z' \in \{x, y\}$. Without loss of generality, let $y' = z$. Since $C_{p,1}$ is a cycle, it follows that there exists a vertex $w \notin \{x, y\}$. Let x', w' be the vertices in $C_{p,2}$ corresponding to x, w in $C_{p,1}$, and let x'', y'', w'' be the vertices in $C_{p,3}$ corresponding to x, y, w in $C_{p,1}$, respectively. Then the vertices x, y, w divide the cycle $C_{p,1}$ into three paths, say P_1, P_2, P_3 . Similarly, the vertices x', z, w' divide the cycle $C_{p,2}$ into three paths, say Q_1, Q_2, Q_3 , and the vertices x'', y'', w'' divide the cycle $C_{p,3}$ into three paths, say R_1, R_2, R_3 . Then the tree induced by the edges in $E(P_2) \cup E(P_3) \cup E(Q_2) \cup \{ww'\}$ and the tree induced by the edges in $E(Q_1) \cup E(R_1) \cup E(R_2) \cup \{xx', x'x''\}$ are two internally disjoint pendant S -Steiner trees in G , where R is a path in some $C_{j,q}$ connecting y and y'' , and hence $\tau(S) \geq 2$.

Suppose that there exist three cycles $C_{p,i}, C_{p,j}, C_{p,k}$ such that $|S \cap V(C_{p,i})| = |S \cap V(C_{p,j})| = |S \cap V(C_{p,k})| = 1$. Without loss of generality, let $|S \cap V(C_{p,1})| = |S \cap V(C_{p,2})| = |S \cap V(C_{p,3})| = 1$. Let y', z' be the vertices corresponding to y, z in $C_{p,1}$, x', z'' be the vertices corresponding to x, z in $C_{p,2}$ and x'', y'' be the vertices corresponding to x, y in $C_{p,3}$. If x, y', z' are distinct vertices in $C_{p,1}$, then the vertices x, y', z' divide the cycle $C_{p,1}$ into three paths P_1, P_2, P_3 , and the vertices x', y, z'' divide the cycle $C_{p,2}$ into three paths Q_1, Q_2, Q_3 , and the vertices x'', y'', z divide the cycle $C_{p,3}$ into three paths R_1, R_2, R_3 . Then the tree induced by the edges in $E(P_1) \cup E(P_2) \cup \{yy', z'z'', zz''\}$ and the tree induced by the edges in $E(Q_1) \cup E(R_3) \cup \{xx', x'x''\}$ are two internally disjoint pendant S -Steiner trees in G , and hence $\tau(S) \geq 2$. Suppose that two of x, y', z' are the same vertex in $C_{p,1}$. Without loss of generality, let $x = y'$. Since $C_{p,1}$ is a cycle, it follows that there exists a vertex v such that $v \notin \{x, z'\}$. Let v', v'' be the vertices in $C_{p,2}, C_{p,3}$ corresponding to v , respectively. Then the vertices x, z', v divide the cycle $C_{p,1}$ into three paths P_1, P_2, P_3 , and the vertices y, z'', v' divide the cycle $C_{p,2}$ into three paths Q_1, Q_2, Q_3 , and the vertices x'', z, v'' divide the cycle $C_{p,3}$ into three paths R_1, R_2, R_3 . Then the tree induced by the edges in $E(P_3) \cup E(Q_3) \cup E(R_2) \cup \{vv', v'v''\}$ and the tree induced by the edges in $E(P_1) \cup E(Q_1) \cup \{z'z'', zz''\}$ are two internally disjoint pendant S -Steiner trees in G , and hence $\tau(S) \geq 2$. Suppose that x, y', z' are the same vertex in $C_{p,1}$. Since $C_{p,1}$ is a cycle, it follows that there exists two vertices u, v such that $u \neq x$ and $v \neq x$. Let u', v' be the vertices in $C_{p,2}$ corresponding to u, v , respectively. Let u'', v'' be the vertices in $C_{p,3}$ corresponding to u, v , respectively. Then the vertices x, u, v divide the cycle $C_{p,1}$ into three paths P_1, P_2, P_3 , and the vertices y, u', v' divide the cycle $C_{p,2}$ into three paths Q_1, Q_2, Q_3 , and the vertices z, u'', v'' divide the cycle $C_{p,3}$ into three paths R_1, R_2, R_3 . Then the tree induced by the edges in $E(P_1) \cup E(Q_1) \cup E(R_1) \cup \{uu', u'u''\}$ and the tree induced by the edges in $E(P_3) \cup E(Q_3) \cup E(R_3) \cup \{vv', v'v''\}$ are two internally disjoint pendant S -Steiner trees in G , and hence $\tau(S) \geq 2$.

From the argument, we conclude that $\tau_3(G) = 2$, and hence $f(n, 3, 2) \leq 2pq = 2n$. From Proposition 2.1, $f(n, 3, 2) \geq 2n$. So $f(n, 3, 2) = 2n$.

(2) Let $G = W_p \square P_2$. From Lemma 3.4, we have $\delta(G) = \kappa(G) = 4$. From Lemma 2.2, $\tau_3(G) \leq \delta(G) - k + 1 = 2$. We only need to show that $\tau_3(G) \geq 2$. It suffices to prove that $\tau(S) \geq 2$ for any $S \subseteq V(G)$ and $|S| = 3$. Let $S = \{x, y, z\}$. Let $W_{p,1}, W_{p,2}$ be the two wheels in G corresponding to W_p . Suppose $S \subseteq V(W_{p,i})$ where $i = 1, 2$. Without loss of generality, let $S \subseteq V(W_{p,1})$. Let x', y', z' be the vertices in $W_{p,2}$ corresponding to x, y, z in $W_{p,1}$, respectively. Since $\kappa(W_{p,1}) = 3$, it follows from Corollary 2.1 that $\tau_3(W_{p,1}) \geq 1$, and hence there exists a pendant S -Steiner tree in $W_{p,1}$, say T . Let T' be the tree in $W_{p,2}$ corresponding to T in $W_{p,1}$. Then the tree induced by the edges in $E(T') \cup \{xx', yy', zz'\}$ and the tree T are two pendant S -Steiner trees in G . Then $\tau(S) \geq 2$. Suppose $|S \cap V(W_{p,1})| = 2$ or $|S \cap V(W_{p,2})| = 2$. Without loss of generality, let $|S \cap V(W_{p,1})| = 2$. Then $|S \cap V(W_{p,2})| = 1$. Let x', y' be the vertices in $W_{p,2}$ corresponding to x, y in $W_{p,1}$, and let z' be the vertex in $W_{p,1}$ corresponding to z in $W_{p,2}$. Let w, w' be the centers of the wheels $W_{p,1}, W_{p,2}$, respectively. Then the tree T_1 induced by the edges in $\{xw, yw, ww', w'z'\}$ is a pendant S -Steiner tree, and the tree T_2 induced by the edges in $\{xx', yy'\}$ is a pendant S -Steiner tree. Since T_1 and T_2 are disjoint, it follows that $\tau(S) \geq 2$. From the arbitrariness of S , we have $\tau_3(G) \geq 2$, and hence $\tau_3(G) = 2$. So $f(n, 3, 2) \leq 5p - 4 = \frac{5n-8}{2}$. From Proposition 2.1, $f(n, 3, 2) \geq 2n$.

(3) From (2) of Theorem 1.1, if n is a prime number, then $2n \leq f(n, 3, 2) \leq 4n - 16$. ■

For $k = 3$ and general ℓ ($3 \leq \ell \leq \frac{n-4}{3}$), we give the upper and lower bound of $f(n, 3, \ell)$.

Proposition 3.3 *Let n, ℓ be two integers with $3 \leq \ell \leq \frac{n-4}{3}$ and $n \geq 15$. Then*

$$\left\lceil \frac{(\ell+2)n}{2} \right\rceil \leq f(n, 3, \ell) \leq (\ell+1)(n-\ell) + 1.$$

Proof. Let $H = \ell K_1 \vee C_{n-\ell}$. Set $V(C_{n-\ell}) = \{y_1, y_2, \dots, y_{n-\ell}\}$. Let G be a graph obtained from H by adding an edge $y_1 y_{\lfloor \frac{n-\ell}{2} \rfloor}$. Then $\delta(G) = \ell + 2$. From Lemma 2.2, $\tau_3(G) \leq \delta(G) - 2 = \ell$. We only need to show $\tau_3(G) \geq \ell$. Set $X = V(G) - V(C_{n-\ell}) = \{x_1, x_2, \dots, x_\ell\}$. For any $S \subseteq V(G)$ and $|S| = 3$, it suffices to show $\tau(S) \geq \ell$. Suppose $S \subseteq V(C_{n-\ell})$. Without loss of generality, let $S = \{y_1, y_2, y_3\}$. Then the trees induced by the edges in $\{x_i y_1, x_i y_2, x_i y_3\}$ ($1 \leq i \leq \ell$) are ℓ internally disjoint pendant S -Steiner trees in G , and hence $\tau(S) \geq \ell$. Suppose $S \subseteq X$. Without loss of generality, let $S = \{x_1, x_2, x_3\}$. Then the trees induced by the edges in $\{y_i x_1, y_i x_2, y_i x_3\}$ ($1 \leq i \leq n - \ell$) are internally disjoint $n - \ell \geq 2\ell + 4$ pendant S -Steiner trees in G , and hence $\tau(S) \geq \ell$. Suppose $|S \cap X| = 2$. Then $|S \cap V(C_{n-\ell})| = 1$. Without loss of generality, let $S = \{x_1, x_2, y_1\}$. Then the trees induced by the edges in $\{y_i x_1, y_i x_2, x_i y_1\}$ ($3 \leq i \leq \ell$), the tree induced by the edges in $\{y_2 x_1, y_2 x_2, y_2 y_1\}$ and the tree induced by the edges in $\{y_{n-\ell} x_1, y_{n-\ell} x_2, y_{n-\ell} y_1\}$ are ℓ internally disjoint pendant S -Steiner trees in G , and

hence $\tau(S) \geq \ell$. Suppose $|S \cap X| = 1$. Then $|S \cap V(C_{n-\ell})| = 2$. Without loss of generality, let $S = \{x_1, y_i, y_j\}$. If $1 \leq i < j \leq \lfloor \frac{n-\ell}{2} \rfloor$, then the trees induced by the edges in $\{x_t y_{\lfloor \frac{n-\ell}{2} \rfloor + t - 1}, y_i x_t, y_j x_t, x_1 y_{\lfloor \frac{n-\ell}{2} \rfloor + t - 1}\}$ ($2 \leq t \leq \ell$) and the tree induced by the edges in $\{y_1 x_{\lfloor \frac{n-\ell}{2} \rfloor}, x_1 y_1\} \cup \{y_r y_{r+1} \mid 1 \leq r \leq i-1\} \cup \{y_s y_{s+1} \mid j \leq s \leq \lfloor \frac{n-\ell}{2} \rfloor - 1\}$ are ℓ internally disjoint pendant S -Steiner trees in G , and hence $\tau(S) \geq \ell$. The same is true for $\lfloor \frac{n-\ell}{2} \rfloor + 1 \leq i < j \leq n - \ell$. Suppose $1 \leq i \leq \lfloor \frac{n-\ell}{2} \rfloor$ and $\lfloor \frac{n-\ell}{2} \rfloor + 1 \leq j \leq n - \ell$. Without loss of generality, let $|j - i| \geq \lceil \frac{n-\ell}{2} \rceil$. Then the trees induced by the edges in $\{x_t y_{i+t}, y_i x_t, y_j x_t, x_1 y_{i+t}\}$ ($2 \leq t \leq \ell$) and the tree induced by the edges in $\{x_1 y_1\} \cup \{y_r y_{r+1} \mid 1 \leq r \leq i-1\} \cup \{y_s y_{s+1} \mid j \leq s \leq \lfloor \frac{n-\ell}{2} \rfloor - 1\}$ are ℓ internally disjoint pendant S -Steiner trees in G , and hence $\tau(S) \geq \ell$. From the argument, we conclude that $\tau(S) \geq \ell$ for any $S \subseteq V(G)$ and $|S| = 3$. So $\tau_3(G) = \ell$ and hence $f(n, 3, \ell) \leq (\ell + 1)(n - \ell) + 1$. ■

Lemma 3.5 [29] *Let G be a connected graph of order $n \geq 6$. Let k be an integer with $3 \leq k \leq n$. Then*

$$\tau_k(G) \leq \kappa(G) - k + 2.$$

For $k = 3$ and general ℓ ($\frac{n-4}{3} \leq \ell \leq \frac{n-r-2}{2}$), we give the upper and lower bound of $f(n, 3, \ell)$.

Proposition 3.4 *Let n, ℓ be two integers with $\frac{n-4}{3} \leq \ell \leq \frac{n-r-2}{2}$ and $n \geq 15$, where $n \equiv r \pmod{\ell + 1}$ and $r \geq 2$. Then*

$$\left\lceil \frac{(\ell + 2)n}{2} \right\rceil \leq f(n, 3, \ell) \leq \left\lfloor \frac{n}{\ell + 1} \right\rfloor (\ell + 1)^2 + (r + 1)(\ell + 1) + r - 1.$$

Proof. Let $F = P_s \circ (\ell + 1)K_1$ where $s = \lfloor \frac{n}{\ell + 1} \rfloor$. Set $P_s = u_1 u_2 \dots u_s$. Let $V((\ell + 1)K_1) = \{v_1, v_2, \dots, v_{\ell + 1}\}$. Let G be the graph obtained from F by adding the vertices $(u_{s+1}, v_1), (u_{s+1}, v_2), \dots, (u_{s+1}, v_r)$ and the edges

$$\{(u_s, v_i)(u_{s+1}, v_j) \mid 1 \leq i \leq \ell + 1, 1 \leq j \leq r\} \cup \{(u_{s+1}, v_j)(u_{s+1}, v_{j+1}) \mid 1 \leq j \leq r - 1\} \\ \cup \{(u_1, v_j)(u_1, v_{j+1}) \mid 1 \leq j \leq \ell\}.$$

Clearly, $\kappa(G) = \ell + 1$. From Lemma 3.5, $\tau_3(G) \leq \kappa(G) - 1 = \ell$. We only need to show $\tau_3(G) \geq \ell$. Let H_1, H_2, \dots, H_s be the copies corresponding to $(\ell + 1)K_1$, and let $H_{s+1} = \{(u_{s+1}, v_j) \mid 1 \leq j \leq r\}$. For any $S \subseteq V(G)$ and $|S| = 3$, it suffices to show $\tau(S) \geq \ell$. Let $S = \{x, y, z\}$.

Suppose that there exists some H_i ($1 \leq i \leq s + 1$) such that $|S \cap V(H_i)| = 3$. Without loss of generality, let $|S \cap V(H_1)| = 3$. Then the trees induced by the edges in $\{x(u_2, v_i), y(u_2, v_i), z(u_2, v_i)\}$ ($1 \leq i \leq \ell + 1$) are $\ell + 1$ internally disjoint pendant S -Steiner trees in G , and hence $\tau(S) \geq \ell + 1$.

Suppose that there exist H_i, H_j such that $|S \cap V(H_i)| = 2$ and $|S \cap V(H_j)| = 1$. Suppose $|j - i| \geq 2$. Note that the subgraph induced by the vertices in $\{(u_r, v_s) \mid i + 1 \leq r \leq j, 1 \leq s \leq \ell + 1\}$ is $(\ell + 1)$ -connected. From Lemma 2.1, there exists a (z, U) -fan, where $U = \{(u_{i+1}, v_s) \mid 1 \leq s \leq \ell + 1\}$, and hence there is a path connecting z and (u_{i+1}, v_s) , say P_s , where $1 \leq s \leq \ell + 1$. Then the trees induced by the edges in $E(P_s) \cup \{(u_{i+1}, v_s)x, (u_{i+1}, v_s)y\}$ are $\ell + 1$ internally disjoint pendant S -Steiner trees, and hence $\tau(S) \geq \ell + 1$. Suppose $|j - i| = 1$. Without loss of generality, let $S \cap V(H_1) = \{x, y\}$ and $|S \cap V(H_2)| = \{z\}$. Let x', y' be the vertices corresponding to x, y in H_2 , z' be the vertex corresponding to z in H_1 . Suppose $z' \notin \{x, y\}$. Without loss of generality, let $\{x, y, z'\} = \{(u_1, v_i) \mid 1 \leq i \leq 3\}$. Then the tree induced by the edges in $\{xx', yx', x'(u_3, v_1), z(u_3, v_1)\}$, the tree induced by the edges in $\{xy', yy', y'z', z'z\}$ and the trees induced by the edges in $\{x(u_2, v_j), y(u_2, v_j), (u_1, v_j)(u_2, v_j), (u_1, v_j)z\}$ ($4 \leq j \leq \ell + 1$) are ℓ internally disjoint pendant S -Steiner trees in G , and hence $\tau(S) \geq \ell$. Suppose $z' \in \{x, y\}$. Without loss of generality, let $z' = y$, $\{x, y\} = \{(u_1, v_i) \mid 1 \leq i \leq 2\}$. Then the tree induced by the edges in $\{xx', yx', x'(u_3, v_1), z(u_3, v_1)\}$ and the trees induced by the edges in $\{x(u_2, v_j), y(u_2, v_j), (u_1, v_j)(u_2, v_j), (u_1, v_j)z\}$ ($3 \leq j \leq \ell + 1$) are ℓ internally disjoint pendant S -Steiner trees in G , and hence $\tau(S) \geq \ell$.

Suppose that there exist H_i, H_j, H_k such that $|S \cap V(H_i)| = |S \cap V(H_j)| = |S \cap V(H_k)| = 1$. Without loss of generality, let $|S \cap V(H_1)| = |S \cap V(H_2)| = |S \cap V(H_3)| = 1$, and $x = (u_1, v_1)$, $y = (u_2, v_1)$, $z = (u_3, v_1)$. Then the trees induced by the edges in $\{x(u_2, v_j), y(u_2, v_j), (u_1, v_j)(u_2, v_j), (u_1, v_j)z\}$ ($2 \leq j \leq \ell + 1$) are ℓ internally disjoint pendant S -Steiner trees in G , and hence $\tau(S) \geq \ell$.

From the above argument, we conclude that $\tau(S) \geq \ell$ for any $S \subseteq V(G)$ and $|S| = 3$. From the arbitrariness of S , we have $\tau_3(G) \geq \ell$, and hence $\tau_3(G) = \ell$. So $f(n, 3, \ell) \leq \lfloor \frac{n}{\ell+1} \rfloor (\ell + 1)^2 + (r + 1)(\ell + 1) + r - 1$. From Proposition 2.1, $f(n, 3, \ell) \geq \lceil \frac{(\ell+2)n}{2} \rceil$, as desired. \blacksquare

The following result is from [28].

Lemma 3.6 [28] *Let G be a connected graph of order n . Then $\tau_3(G) = n - 5$ if and only if \bar{G} is a subgraph of one of the following graphs.*

- $C_i \cup C_j \cup (n - i - j)K_1$ ($i = 3, 4, j = 3, 4$);
- $C_i \cup \lfloor \frac{n-i}{2} \rfloor K_2$ ($i = 3, 4$);
- $P_5 \cup \lfloor \frac{n-5}{2} \rfloor K_2$;
- $C_i \cup (n - i)K_1$ ($i = 5, 6, 7$).

Proof of Theorem 1.3: From (3) of Theorem 1.1, we have $f(n, 3, 0) = n - 1$. From (4) of Theorem 1.1, we have $f(n, 3, 1) = \lceil \frac{3n}{2} \rceil$. From Proposition 3.2, $f(n, 3, 2) = 2n$ for $n = pq$

and $p, q \geq 3$; $2n \leq f(n, 3, 2) \leq \frac{5n}{2}$ for $n = 2p$; $2n \leq f(n, 3, 2) \leq 4n - 16$ if n is a prime number. From Proposition 3.3, $\lceil \frac{(\ell+2)n}{2} \rceil \leq f(n, 3, \ell) \leq (\ell+1)(n-\ell) + 1$ for $3 \leq \ell \leq \frac{n-4}{3}$; From Proposition 3.4, $\lceil \frac{(\ell+2)n}{2} \rceil \leq f(n, 3, \ell) \leq \lfloor \frac{n}{\ell+1} \rfloor (\ell+1)^2 + (r+1)(\ell+1) + r - 1$ for $\frac{n-1}{3} \leq \ell \leq \frac{n-r-2}{2}$ where $n \equiv r \pmod{\ell}$ and $r \geq 2$. From (5) of Theorem 1.1, we have $\lceil \frac{(\ell+2)n}{2} \rceil \leq f(n, 3, \ell) \leq (\ell+2)(n-\ell-2)$ for $\frac{n-1}{3} \leq \ell \leq \frac{n-r-2}{2}$ where $n \equiv r \pmod{\ell}$ and $r = 0, 1$, or $\frac{n-r-2}{2} \leq \ell \leq \frac{n-4}{2}$. From (6) of Theorem 1.1, we have $\lceil \frac{(\ell+2)n}{2} \rceil \leq f(n, 3, \ell) \leq (\ell+2)(n-\ell-2) + \binom{\ell+2}{2}$ for $\frac{n-2}{2} \leq \ell \leq n-6$. From Lemma , we have $f(n, 3, n-5) = \binom{n}{2} - \lfloor \frac{n-4}{2} \rfloor$. From (1) and (2) of Theorem 1.1, $f(n, 3, n-4) = \binom{n}{2} - 2$ and $f(n, 3, n-3) = \binom{n}{2}$. ■

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